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On almost rigid rotations. Part 2

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The dynamical properties of a fluid, occupying the space between two concentric rotating spheres, are considered, attention being focused on the case where the angular velocities of the spheres are only slightly different and the Reynolds number R of the flow is large. It is found that the flow properties differ inside and outside a cylinder $\mathscr{C}$, circumscribing the inner sphere and having its generators parallel to the axis of rotation. Outside $\mathscr{C}$ the fluid rotates as if rigid with the angular velocity of the outer sphere. Inside $\mathscr{C}$ the fluid rotates with an angular velocity intermediate to the angular velocities of the two spheres and determined by the condition that the flux of fluid into the boundary layer of the faster-rotating sphere is equal to the flux out of the boundary layer of the slower-rotating sphere at the same distance from the axis. The return of fluid is effected by a shear layer near $\mathscr{C}$ and we show that it has a complicated structure for it can be divided into three separate layers, two outer ones, of thickness $\sim R^{-\frac{2}{7}}$ and $\sim R^{-\frac{1}{4}}$, and an inner layer of thickness $\sim R^{-\frac{1}{3}}$.

1. Introduction

The study of shear layers of finite length, near surfaces parallel to the axis of rotation of a fluid, was initiated by Proudman (1956). They arose in his study of the almost rigid rotation of a viscous fluid between two concentric spheres, rotating about a common axis $\mathbf{l}$ with angular velocities Ω and $\Omega(1+\epsilon)$ where $\epsilon \ll 1$. He found that the cylinder $\mathscr{C}$, circumscribing the inner sphere and having its generators parallel to $\mathbf{l}$, separated out regions with different secondary-flow properties. Outside $\mathscr{C}$ the fluid rotates as if solid with the angular velocity $\Omega(1+\epsilon)$ of the outer sphere. Inside $\mathscr{C}$ the angular velocity is almost uniform, the departure from the uniform state being determined by a balance of fluid expelled from the Ekman layer on the inner, and supposedly more slowly rotating, sphere to the fluid drawn into the Ekman layer on the outer and faster-rotating sphere. Such a balance is necessary for the fluid can only move, outside the boundary layers, on cylinders parallel to $\mathscr{C}$. A return circuit for this fluid is provided by a shear layer on $\mathscr{C}$, which must also adjust the discontinuity in the angular velocity across $\mathscr{C}$. Proudman speculated on the structure of this shear layer but without coming to any firm conclusions.

The main difficulties with this shear layer are best understood in terms of the Reynolds number $R = \Omega a^2/\nu$ of the flow, where a is the radius of the inner sphere and ν is the kinematic viscosity, and which we suppose is large. Then the rate at which fluid must be fed into the Ekman layers $\sim R^{-\frac{1}{2}}$ and, if the viscous and inertia terms are comparable in both equations of motion, the thickness of the shear layer $\sim R^{-\frac{1}{3}}$, while the axial and the perturbed azimuthal velocities are of the same order of magnitude. Hence, in order to feed the Ekman layers, the perturbed azimuthal velocity $\sim R^{-\frac{1}{6}}$ while the discontinuity in angular velocity which must also be smoothed out $\sim R^0$. On the other hand, if we make the perturbed azimuthal velocity $\sim R^0$, the flux of fluid into the Ekman layers must be $R^{-\frac{1}{3}}$.

Alternatively, one might balance the viscous and inertia terms in one equation but not in the other. In this case the shear layer has thickness $\sim R^{-\frac{1}{4}}$ and the perturbed azimuthal velocity can be chosen $\sim R^0$ with the axial velocity $\sim R^{-\frac{1}{4}}$ so that the flux of fluid into the Ekman layer $\sim R^{-\frac{1}{2}}$, as desired. On the other hand the detailed structure of this shear layer appeared to be indeterminate.

Most of these difficulties were removed in two subsequent papers. In the first, with the same title as the present paper, Stewartson (1957) studied a related problem, but with a different geometry, namely, the flow between two parallel, differentially rotating, coaxial disks. The advantage of this geometry is that a full solution of the linearized equations can be formally written down, from which the limit structures as $R \to \infty$ can be deduced. It was found that both types of shear layer envisaged by Proudman occur, the outer one (of thickness $\sim R^{-\frac{1}{4}}$) smoothing out the discontinuity in the angular velocity, but *not* in its second derivative, and the inner (of thickness $\sim R^{-\frac{1}{3}}$) completing the smoothing of the angular velocity and contributing to the feeding of the Ekman layers.

Although the structure of the shear layer is clear for this special geometry, the formulation of appropriate boundary conditions, by which the structure could be elucidated in general, was first made by Jacobs (1964). He pointed out that since the Ekman boundary layer is much thinner than the shear layer (thickness $\sim R^{-\frac{1}{2}}$ as against $\sim R^{-\frac{1}{3}}$) the compatibility condition on the fluid velocities just outside the Ekman layer must hold not only outside the shear layer but inside it too. Such a condition occurs at both ends of the shear layer in the rotating disks problem and completes the specification of the layer.

There are two points in connexion with Jacobs's paper which should be noted. First, Jacobs claimed that the shear layer can only exist if the flow outside satisfies a certain relation. If true this implies that, like the Ekman layer, the shear layer exerts a control on fluid properties outside it. Secondly, his compatibility conditions at the ends of the shear layer were found on the assumption that the corresponding Ekman layer makes a non-zero angle with the shear layer. This requirement suggests that his ideas cannot be applied to Proudman's problem because the shear layer almost touches the inner sphere.

I disagree with Jacobs's first contention and believe that the shear layer is derivative; i.e. it is determined by the local properties of the inviscid and Ekman flows around it, not vice versa. As illustration we consider in the present paper the problem first studied by Proudman and we shall show that the main elements of the structure of the shear layer follow in this way. The main purpose of the paper, however, is to show that Jacobs's second restriction is not necessary and that even when the Ekman layer is almost parallel to the shear layer the compatibility condition can still be applied.

It is found that the equations governing the Ekman layer on the inner sphere are valid until one is within a distance $\sim R^{-\frac{2}{5}}$ of $\mathscr{C}$. Since the shear layer has a thickness $\sim R^{-\frac{1}{3}}$ at least, the region of invalidity of the Ekman layer occupies a negligible part of the shear layer. The shear layer itself may be divided into three parts. There are two outer layers, one outside $\mathscr{C}$ of thickness $\sim R^{-\frac{1}{4}}$ in which the majority of fluid is transferred from one Ekman layer to the other, and one inside $\mathscr{C}$, of thickness $\sim R^{-\frac{2}{7}}$, whose main purpose is to remove a singularity in the gradient of the azimuthal velocity. These layers are separated by an inner shear layer, of thickness $\sim R^{-\frac{1}{3}}$, in which the discontinuity in the *second* derivative of the azimuthal velocity and the consequent discontinuity in the velocity,

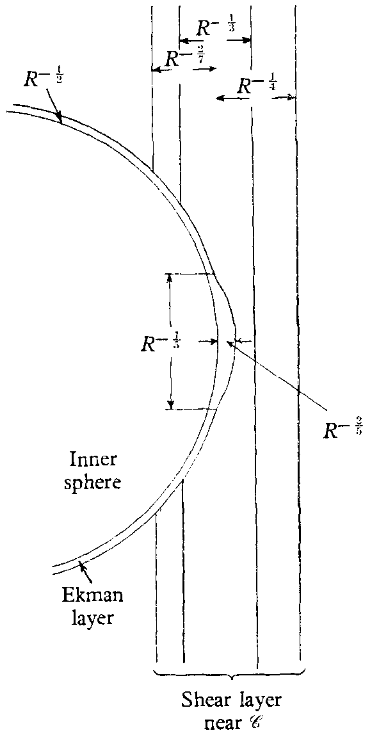

FIGURE 1. Schematic drawing (not to scale) of the intersection region of the Ekman layer near the inner sphere and the shear layer near $\mathscr{C}$.

radially out from the axis, are smoothed out. In this layer the azimuthal velocity $\sim R^{-\frac{1}{28}}$, which is small, in contrast with the problem of the two rotating disks, where the perturbed azimuthal velocity in the inner shear layer $\sim R^{0}$. The fact that it is not zero on $\mathscr{C}$, however, means that the velocities in the inner shear layer develop singularities as the inner sphere is approached and it is shown that these are consistent with the structure of the layer formed by the merging of the Ekman and shear layers and implies that the fluid velocity in this layer $\sim R^{-\frac{1}{28}}$. A schematic drawing of the intersection region of the Ekman layer near the inner sphere and the shear layer near $\mathscr{C}$ is shown in figure 1.

Proudman included a discussion of the conditions necessary for the linearization of the governing equation which all investigators have used. Applying his arguments to the detailed flow properties found in this paper shows that as before the criterion is $\epsilon R^{\frac{1}{3}} \ll 1$.

2. The statement of the problem

Following Proudman (1956), let the radii of the inner and outer spheres be a and αa respectively and let the corresponding angular velocities be Ω and $\Omega(1+\epsilon)$ where ϵ is very small (but may be positive or negative). Let (ar, θ, ϕ) denote spherical polar co-ordinates in which the line $\theta = 0$ coincides with the axis of rotation and let $(a\Omega u, a\Omega v, a\Omega w)$ be the corresponding components of velocity (see figure 2).

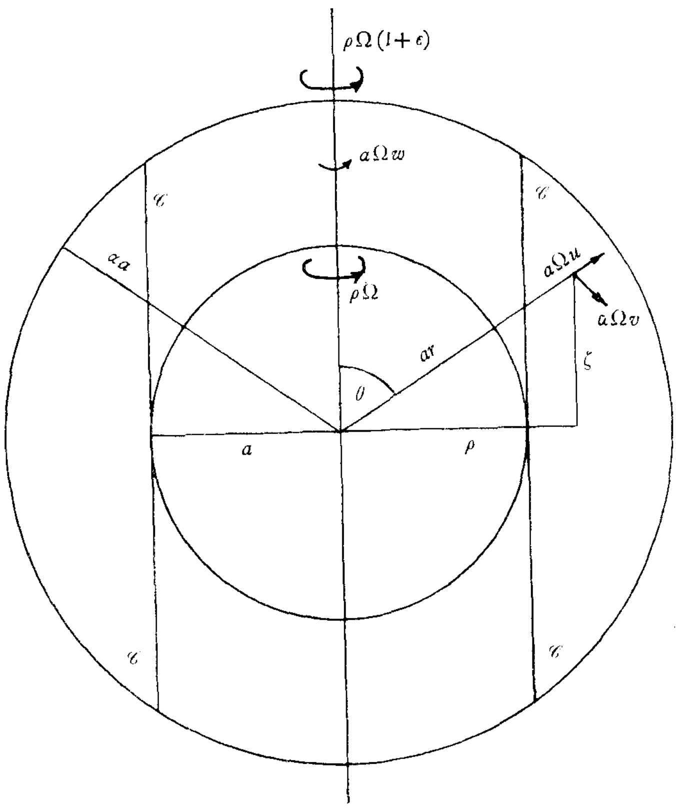

FIGURE 2. Notation.

By symmetry, all dynamical variables must be independent of ϕ and the velocity components may be expressed in terms of two functions ψ', χ', viz.:

$$u = \frac{1}{r^2 \sin\theta}\frac{\partial \psi'}{\partial \theta}, \quad v = -\frac{1}{r\sin\theta}\frac{\partial \psi'}{\partial r}, \quad w = \frac{\chi'}{r\sin\theta}, \tag{2.1}$$

using the equation of continuity. The equations of momentum then reduce to

$$\frac{2\chi'}{r^2\sin^2\theta}\left[\frac{\partial \chi'}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \chi'}{\partial \theta}\sin\theta\right] - \frac{1}{r^2\sin\theta}\frac{\partial(\psi', D^2\psi')}{\partial(r, \theta)} + \frac{2D^2\psi'}{r^2\sin^2\theta}\left[\frac{\partial \psi'}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \psi'}{\partial \theta}\sin\theta\right] = \frac{1}{R}D^4\psi', \tag{2.2}$$

and
$$-\frac{1}{r^2\sin\theta}\frac{\partial(\psi',\chi')}{\partial(r,\theta)}=\frac{1}{R}D^2\chi', \tag{2.3}$$
where
$$D^2=\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$
and R is the Reynolds number of the flow, defined by
$$R=a^2\Omega/\nu, \tag{2.4}$$
ν being the kinematic viscosity.

The boundary conditions for these equations are that
$$\partial\psi'/\partial r=\psi'=0,\quad \chi'=\sin^2\theta\quad\text{at}\quad r=1, \tag{2.5}$$
and
$$\partial\psi'/\partial r=\psi'=0,\quad \chi'=\alpha^2(1+\epsilon)\sin^2\theta\quad\text{at}\quad r=\alpha.$$
When $\epsilon=0$ this problem clearly has the exact solution
$$\psi'=0,\quad \chi'=r^2\sin^2\theta \tag{2.6}$$
and, when ϵ is sufficiently small, we shall assume that it is legitimate to write
$$\psi'=\epsilon\psi,\quad \chi'=r^2\sin^2\theta+\epsilon\chi, \tag{2.7}$$
and to neglect squares of ϵ. The dynamical equations then reduce to
$$2\left(\frac{\partial\chi}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial\chi}{\partial\theta}\sin\theta\right)=\frac{1}{R}D^4\psi, \tag{2.8}$$
$$-2\left(\frac{\partial\psi}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right)=\frac{1}{R}D^2\chi, \tag{2.9}$$
and the boundary conditions to
$$\psi=\partial\psi/\partial r=0,\quad \chi=0\quad\text{at}\quad r=1 \tag{2.10}$$
and
$$\psi=\partial\psi/\partial r=0,\quad \chi=\alpha^2\sin^2\theta\quad\text{at}\quad r=\alpha.$$

It also proves convenient to write down these equations in cylindrical polar co-ordinates (ρ,ϕ,ζ) where
$$\rho=r\sin\theta,\quad \zeta=r\cos\theta. \tag{2.11}$$
We obtain
$$(1/R)\,D^4\psi=2\,\partial\chi/\partial\zeta, \tag{2.12}$$
$$(1/R)\,D^2\chi=-2\,\partial\psi/\partial\zeta, \tag{2.13}$$
where
$$D^2=\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial\zeta^2}. \tag{2.14}$$

3. The boundary layers on the spheres

As Proudman has pointed out, when the Reynolds number is large the viscous terms may reasonably be neglected except in the neighbourhood of certain singular surfaces. Such surfaces are the boundaries of the spheres and the cylinder $\mathscr{C}$ circumscribing the inner sphere and having its generators parallel to the axis of rotation. Apart from these regions we have, directly from (2.12), (2.13) on setting $R = \infty$,

$$\chi = \chi_0(\rho), \quad \psi = \psi_0(\rho). \tag{3.1}$$

Thus, in the main body of the fluid there is no radial motion, while the axial and azimuthal components of velocity are independent of ζ. This solution however fails to satisfy the boundary conditions on both spheres simultaneously and consequently there must be boundary layers set up in the neighbourhood of one or both of the spheres. Suppose that one is set up near the sphere $r = 1$. Then in it $\partial/\partial r \gg 1$, while along the majority of its length $\partial/\partial\theta \sim 1$ since it is partly controlled by (3.1). Hence (2.8), (2.9) reduce to

$$\partial^4\psi/\partial r^4 = 2R\cos\theta\,\partial\chi/\partial r, \quad \partial^2\chi/\partial r^2 = -2R\cos\theta\,\partial\psi/\partial r. \tag{3.2}$$

The boundary conditions at the sphere $r = 1$ are

$$\psi = \partial\psi/\partial r = 0, \quad \chi = 0, \tag{3.3}$$

while on leaving the boundary layer the solution must tend to (3.1). Proudman showed that this is only possible if

$$\chi_0 = 2(R\cos\theta)^{\frac{1}{2}}\psi_0 \tag{3.4}$$

and then, in the boundary layer,

$$\psi = \psi_0[1-e^{-\eta}(\cos\eta+\sin\eta], \quad \chi = \chi_0[1-e^{-\eta}\cos\eta], \tag{3.5}$$

where
$$\eta = (r-1)(R\cos\theta)^{\frac{1}{2}}.$$

Thus the azimuthal and axial components of velocity in the main core of the fluid, outside the shear layers, are not independent but satisfy a relation which may be put in the form

$$\chi_0(\rho) = 2R^{\frac{1}{2}}(1-\rho^2)^{\frac{1}{4}}\psi_0(\rho). \tag{3.6}$$

In a similar way, from a study of the boundary layer on the outer sphere we find that

$$\rho^2-\chi_0(\rho) = 2R^{\frac{1}{2}}(1-\rho^2/\alpha^2)^{\frac{1}{4}}\psi_0(\rho) \tag{3.7}$$

and hence
$$\psi_0(\rho) = \frac{\rho^2}{2R^{\frac{1}{2}}}[(1-\rho^2/\alpha^2)^{\frac{1}{4}}+(1-\rho^2)^{\frac{1}{4}}]^{-1}, \tag{3.8}$$

$$\chi_0(\rho) = \rho^2(1-\rho^2)^{\frac{1}{4}}[(1-\rho^2/\alpha^2)^{\frac{1}{4}}+(1-\rho^2)^{\frac{1}{4}}]^{-1} \tag{3.9}$$

provided $\rho < 1$. Inside $\mathscr{C}$ therefore the leading terms in the expansion of the velocities in the core about the point $R^{-1} = 0$ are determined from the boundary layers on the two spheres. Outside $\mathscr{C}$ (3.6) is not relevant. On the basis of a linearized theory the possibility of a radial jet near the surface $\zeta = 0$ can be ruled out (Proudman 1956) and the appropriate additional condition needed is that

$$\psi_0 = 0 \quad \text{at} \quad \zeta = 0, \quad \rho > 1, \tag{3.10}$$

which follows from symmetry considerations. In consequence

$$\chi_0 \equiv \rho^2, \quad \psi_0 \equiv 0 \quad \text{in} \quad \rho > 1, \tag{3.11}$$

so that the fluid outside $\mathscr{C}$ rotates as if solid with the angular velocity of the outer sphere.

Taking $\epsilon > 0$ for convenience, the secondary flow inside $\mathscr{C}$ may be described as follows. Fluid is drawn into the boundary layer on the outer (and faster-rotating) sphere and, once in it, moves away from the axis of rotation towards $\mathscr{C}$. At $\mathscr{C}$ it is presumably turned round, moves down $\mathscr{C}$ in a shear layer, turns round near the circle of contact between $\mathscr{C}$ and the inner sphere and moves back towards the axis in the boundary layer of the inner sphere. As it moves back it continually loses fluid which moves very slowly parallel to the axis of rotation towards the outer sphere and completes the circuit. The shear layer near $\mathscr{C}$ is also necessary to smooth out the discontinuity in the angular velocity for, from (3.9), (3.11),

$$\chi_0 \to 0 \quad \text{as} \quad \rho \to 1-, \qquad \chi_0 \to 1 \quad \text{as} \quad \rho \to 1+. \tag{3.12}$$

Before studying it, however, it is convenient to discuss the properties of the boundary layer on the inner sphere in the neighbourhood of $\rho = 1$. The simple form (3.5) taken by this boundary layer breaks down as $\theta \to \frac{1}{2}\pi$ because then $\partial/\partial r$ formally tends to zero. The essential conditions that must be satisfied in order to reduce (2.8), (2.9) to (3.2) are that

$$\partial/\partial r \gg \partial/\partial\theta \quad \text{and} \quad \cos\theta\,\partial/\partial r \gg \partial/\partial\theta. \tag{3.13}$$

Taking $\partial/\partial r \sim (R\cos\theta)^{\frac{1}{2}}$ and $\partial/\partial\theta \sim (\frac{1}{2}\pi - \theta)^{-1}$, the second condition is seen to be of greater significance and we can expect a modification in the equations to be necessary when

$$\cos\theta(R\cos\theta)^{\frac{1}{2}} \sim (\tfrac{1}{2}\pi - \theta)^{-1}, \quad \text{i.e.} \quad \tfrac{1}{2}\pi - \theta \sim R^{-\frac{1}{5}}. \tag{3.14}$$

The re-scaling necessary to make the equations formally independent of R when $\frac{1}{2}\pi - \theta \sim R^{-\frac{1}{5}}$ is

$$\chi = \chi^*, \quad \psi = R^{-\frac{3}{5}}\psi^*, \quad \tfrac{1}{2}\pi - \theta = R^{-\frac{1}{5}}\Theta, \quad r = 1 + \lambda R^{-\frac{2}{5}}. \tag{3.15}$$

It is noted, for future reference, that in this region we can write

$$\zeta = R^{-\frac{1}{5}}\zeta^*, \quad \rho = 1 + R^{-\frac{2}{5}}y^*,$$

where

$$\zeta^* = \Theta, \quad y^* = \lambda - \tfrac{1}{2}\Theta^2. \tag{3.16}$$

The equations governing χ^*, ψ^* in the limit $R \to \infty$ may then be written in the alternative forms

$$\frac{\partial^4\psi^*}{\partial\lambda^4} = 2\left[\Theta\frac{\partial\chi^*}{\partial\lambda} + \frac{\partial\chi^*}{\partial\Theta}\right], \quad \frac{\partial^2\chi^*}{\partial\lambda^2} = -2\left[\Theta\frac{\partial\psi^*}{\partial\lambda} + \frac{\partial\psi^*}{\partial\Theta}\right] \tag{3.17}$$

or

$$\frac{\partial^4\psi^*}{\partial y^{*4}} = 2\frac{\partial\chi^*}{\partial\zeta^*}, \quad \frac{\partial^2\chi^*}{\partial y^{*2}} = -2\frac{\partial\psi^*}{\partial\zeta^*}.\dagger \tag{3.18}$$

The appropriate boundary conditions for these equations will be briefly discussed in § 7 below.

† [*Note added in proof.*] An equivalent set of equations was written down by Roberts & Stewartson (1963) in their study of the stability of Maclaurin spheroids and by Carrier (1965). Carrier also indicated a method of solution by iteration, which depended on the terminal region being bounded in λ and is not appropriate here. A solution for a region in which λ is unbounded has recently been obtained, in numerical form, by J. M. Dowden (unpublished).

4. The shear layer: general remarks

In the linearized theory, the fluid moving parallel to the axis of rotation, from the Ekman boundary layer on the inner sphere to the Ekman layer on the outer, must return via a shear layer near $\mathscr{C}(\rho = 1)$. This shear layer must also smooth out the discontinuity in the azimuthal velocity of the fluid across $\mathscr{C}$, according to the theory in § 3. In such a layer differentiation with respect to ρ can be expected to have a magnifying effect when $R \gg 1$, unlike differentiation with respect to ζ. Hence the governing equations (2.12), (2.13) simplify to

$$\partial^4\psi/\partial\rho^4 = 2R\,\partial\chi/\partial\zeta, \quad \partial^2\chi/\partial\rho^2 = -2R\,\partial\psi/\partial\zeta \tag{4.1}$$

in this layer. Some of the appropriate boundary conditions for these equations can be written down in comparison with (3.8)–(3.11). We have, in fact,

$$\psi \to 0, \quad \chi \to 1 \tag{4.2a}$$

as $\rho - 1 \to \infty$ on the scale of the shear layer. It will be shown later that this is equivalent to requiring $R^{\frac{1}{2}}\psi$, $\chi - 1$ to be small when $(\rho - 1)R^{\frac{1}{4}} \gg 1$, but of course $\rho - 1$ being small. Further, retaining leading terms only,

$$\psi \to \frac{1}{2R^{\frac{1}{2}}}\left(1 - \frac{1}{\alpha^2}\right)^{-\frac{1}{4}}, \quad \chi \approx (2(1-\rho))^{\frac{1}{4}}\left(1 - \frac{1}{\alpha^2}\right)^{-\frac{1}{4}} \tag{4.2b}$$

as $\rho - 1 \to -\infty$ on the scale of the shear layer. Here we have in mind that (4.2b) holds when $(1-\rho)R^{\frac{2}{7}} \gg 1$.

These boundary conditions were written down by Proudman but in themselves, as he pointed out, are not sufficient to solve the equations. In addition conditions on χ, ψ at $\zeta = 0$, $(\alpha^2 - 1)^{\frac{1}{2}}$ are needed. For a related problem, involving two parallel, coaxially rotating disks Stewartson (1957) was able to complete the description of the flow in the shear layer because, fortuitously, a full solution of the basic equations (2.12), (2.13) is available. The way to determine the extra conditions at $\zeta = 0$, $(\alpha^2 - 1)^{\frac{1}{2}}$ has recently been pointed out by Jacobs (1964). He observed that the shear layer has thickness of order $R^{-\frac{1}{3}}$ at least, while the Ekman boundary layers are of thickness $R^{-\frac{1}{2}}$. Hence, over almost the entire width of the shear layer, the magnifying effect of $\partial/\partial\rho$, although large, is still smaller than the magnifying effect of $\partial/\partial r$ in the Ekman layer and can therefore be neglected. It follows that the compatibility condition (3.7) of the Ekman boundary layer must be satisfied and we have

$$1 - \chi = 2R^{\frac{1}{2}}[(\alpha^2 - 1)/\alpha^2]^{\frac{1}{4}}\psi \quad \text{at} \quad \zeta = (\alpha^2 - 1)^{\frac{1}{2}}. \tag{4.3}$$

Jacobs restricted his observations to Ekman layers on surfaces intersecting a shear layer at positive angles, which is not the case with the intersection of the shear layer at $\mathscr{C}$ and the inner sphere. It is legitimate however to extend his idea to such an intersection because the compatibility condition (3.6), at the outer edge of the Ekman layer, holds so long as $\Theta \gg 1$, i.e.

$$1 - \rho \gg R^{-\frac{2}{5}} \tag{4.4}$$

from (3.16). In the shear layer $1 - \rho \sim R^{-\frac{1}{3}}$ and satisfies this criterion. Hence, for the leading terms in the expansion of ψ, χ in descending powers of R is

$$\chi = 2R^{\frac{1}{2}}[2(1-\rho)]^{\frac{1}{4}}\psi \quad \text{at} \quad \zeta = 0, \tag{4.5}$$

provided $\rho < 1$. Finally

$$\psi = 0 \quad \text{at} \quad \zeta = 0, \quad \rho > 1, \tag{4.6}$$

from symmetry considerations and because a linearized theory forbids a radial jet to spread out from the equatorial plane of the inner sphere parallel to the ζ direction.

It is now established that *usually* this shear layer must be subdivided into three layers: an inner layer of thickness $O(R^{-\frac{1}{3}})$ in which all terms of (4.1) are of the same order and the changes in $\chi \ll 1$, and on either side of it an outer layer of thickness $O(R^{-\frac{1}{4}})$ in which the dependence of χ on ζ may be neglected and in which $\chi \sim 1$. The main purpose of the two outer layers is to adjust the azimuthal velocity χ and this induces a secondary axial flow with a velocity $O(R^{-\frac{1}{4}})$. The inner layer acts partly as a return pipe for this axial motion and partly to carry fluid from one Ekman layer to the other.

Broadly speaking the same is true here but some modification is necessary in the light of (4.1) and will be discussed in the next section.

5. The outer layers

In these outer layers $|(\rho-1)R^{\frac{1}{3}}| \gg 1$, and, following Jacobs, we suppose that in them $\partial\chi/\partial\zeta$ may be neglected. It follows that χ is a function of ρ only and consequently

$$\psi = -\frac{\zeta}{2R}\frac{\partial^2\chi}{\partial\rho^2}+f(\rho), \tag{5.1}$$

where $f(\rho)$ is a function of ρ to be found. First of all we take $\rho > 1$ so that (4.6) holds at $\zeta = 0$. Then $f(\rho) \equiv 0$ and hence, from the boundary condition (4.3) at $\zeta = (\alpha^2-1)^{\frac{1}{2}}$,

$$\frac{\sqrt{(\alpha^2-1)}}{2R}\frac{\partial^2\chi}{\partial\rho^2} = \frac{\chi-1}{2R^{\frac{1}{2}}}\left[\frac{\alpha^2}{\alpha^2-1}\right]^{\frac{1}{4}}. \tag{5.2}$$

The solution of this differential equation which satisfies (4.2*b*) is

$$\chi = 1-A\exp-\{R^{\frac{1}{4}}\alpha^{\frac{1}{4}}(\rho-1)/(\alpha^2-1)^{\frac{3}{8}}\}, \tag{5.3a}$$

from which

$$\psi = (\zeta\alpha^{\frac{1}{2}}A)/\{2(\alpha^2-1)^{\frac{3}{4}}R^{\frac{1}{2}}\}\exp-\{R^{\frac{1}{4}}\alpha^{\frac{1}{4}}(\rho-1)/(\alpha^2-1)^{\frac{3}{8}}\}, \tag{5.3b}$$

where A is (at present) an arbitrary constant. It may now be confirmed from (4.1) that $\partial\chi/\partial\zeta = O(R^{-\frac{1}{2}})$ and is negligible as assumed. It is noted that this outer boundary layer is of thickness $O(R^{-\frac{1}{4}})$ and, apart from adjusting the azimuthal component of velocity, also transports fluid from the boundary layer on the faster-rotating sphere towards the boundary layer on the other.

Secondly, we take $\rho < 1$ so that (4.5) holds at $\zeta = 0$. Then

$$f(\rho) = \frac{\chi}{2R^{\frac{1}{2}}[2(1-\rho)]^{\frac{1}{4}}} \tag{5.4}$$

and hence, using the condition (4.3) at $\zeta = (\alpha^2-1)^{\frac{1}{2}}$

$$\frac{\sqrt{(\alpha^2-1)}}{2R}\frac{\partial^2\chi}{\partial\rho^2} = \frac{\chi-1}{2R^{\frac{1}{2}}}\left[\frac{\alpha^2}{\alpha^2-1}\right]^{\frac{1}{4}}+\frac{\chi}{2R^{\frac{1}{2}}[2(1-\rho)]^{\frac{1}{4}}}. \tag{5.5}$$

A solution is required which satisfies (4.2*a*) and this can formally be written as an expansion in descending powers of R whose coefficients are functions of

$$s = (1-\rho)[R^2/2(\alpha^2-1)^2]^{\frac{1}{7}} \tag{5.6}$$

only. We obtain

$$\chi = B\mathscr{F}(s) + 2^{\frac{2}{7}}\alpha^{\frac{1}{2}}(\alpha^2-1)^{-\frac{5}{28}}R^{-\frac{1}{14}}\mathscr{G}(s) + O(BR^{-\frac{1}{14}}, R^{-\frac{1}{7}}), \tag{5.7}$$

where B is an arbitrary constant, later shown to be $O(R^{-\frac{1}{28}})$, and

$$\mathscr{F}''(s) - s^{-\frac{1}{4}}\mathscr{F}(s) = 0, \quad \text{with} \quad \mathscr{F}(0) = 1, \quad \mathscr{F}(\infty) = 0, \tag{5.8}$$

$$\mathscr{G}'' - s^{-\frac{1}{4}}\mathscr{G} = -1 \quad \text{with} \quad \mathscr{G}(0) = 0, \quad \mathscr{G} - s^{\frac{1}{4}} \quad \text{bounded as} \quad s \to \infty. \tag{5.9}$$

It is noted in passing that

$$\mathscr{F}(s) = 2(\tfrac{4}{7})^{\frac{11}{7}}s^{\frac{1}{2}}K_{\frac{4}{7}}(\tfrac{8}{7}s^{\frac{7}{8}})/(\tfrac{4}{7})!, \tag{5.10}$$

where $K_{\frac{4}{7}}$ is the Bessel function, of order $\frac{4}{7}$, of the second kind and with imaginary argument, so that

$$\mathscr{F}'(0) = -(\tfrac{4}{7})^{\frac{8}{7}}(-\tfrac{4}{7})!/(\tfrac{4}{7})!. \tag{5.11}$$

The corresponding value of ψ is

$$\psi = \frac{(\alpha^2-1)^{\frac{1}{2}} - \zeta}{2R^{\frac{1}{2}}(\alpha^2-1)^{\frac{1}{2}}}[2(1-\rho)]^{-\frac{1}{4}}\chi + \frac{\zeta(1-\chi)\alpha^{\frac{1}{2}}}{2R^{\frac{1}{2}}(\alpha^2-1)^{\frac{3}{4}}}. \tag{5.12}$$

In contrast to the other layer this one is of thickness $O(R^{-\frac{2}{7}})$, and the leading term is independent of the condition at $\zeta = (\alpha^2-1)^{\frac{1}{2}}$. There is also a transport of fluid *away* from the inner sphere although none of it reaches the outer sphere, for the value of ψ at $\zeta = (\alpha^2-1)^{\frac{1}{2}}$ is actually slightly less when $s \sim 1$ than when $s \gg 1$.

6. The inner layer

The solution given in the previous section must fail at $\rho = 1$, in some sense, because the second derivatives of χ with respect to ρ and the values of ψ are not continuous there. An inner layer is therefore required to adjust these two quantities and it must be $O(R^{-\frac{1}{3}})$ in thickness. However, in the solutions given, χ and $\partial\chi/\partial y$ are not continuous either, unless A and B satisfy certain conditions, and we shall now show that, if these conditions are not satisfied, then no solution of the inner layer can be found.

First, suppose that, according to the solution in the outer shear layers, the inner layer has to adjust a discontinuity in χ whose order of magnitude is R^β $(\beta < 0)$. This would be the case if $1 - A - B = O(R^\beta)$. In the inner shear layer we then write

$$\rho - 1 = R^{-\frac{1}{3}}v, \quad \chi = B + R^\beta\hat{\chi}_1 + \ldots \tag{6.1}$$

the dots here and subsequently denoting that terms of higher order in R^{-1} are omitted. It follows from the governing equations that ψ may be written as

$$\psi = R^{\beta-\frac{1}{3}}\hat{\psi}_1 + \ldots, \tag{6.2}$$

where $\hat{\chi}_1$, $\hat{\psi}_1$ are independent of R. The boundary condition at $\zeta = (\alpha^2-1)^{\frac{1}{2}}$ (4.3) then becomes

$$B - 1 + R^\beta\hat{\chi}_1 = -2R^{\beta+\frac{1}{6}}[(\alpha^2-1)/\alpha^2]^{\frac{1}{4}}\hat{\psi}_1, \tag{6.3}$$

and reduces to

$$\hat{\psi}_1 = 0 \tag{6.4}$$

in the limit $R \to \infty$ provided that $\beta > -\frac{1}{6}$. The boundary condition at $\zeta = 0$, $\rho < 1$ (4.5) becomes

$$B + R^{\beta}\hat{\chi}_1 = 2[-2\upsilon]^{\frac{1}{4}}\hat{\psi}_1 R^{\beta+\frac{1}{12}} \tag{6.5}$$

and reduces to (6.4) if $\beta > -\frac{1}{12}$. The boundary condition at $\zeta = 0$, $\rho > 1$ (4.6) is already the same as (6.4).

Now integrate

$$\partial^2\chi/\partial\rho^2 = -2R\,\partial\psi/\partial\zeta \tag{6.6}$$

with respect to ζ, from 0 to $(\alpha^2-1)^{\frac{1}{2}}$, and, in terms of υ, we get for the leading terms

$$\frac{\partial^2}{\partial\upsilon^2}\int_0^{(\alpha^2-1)^{\frac{1}{2}}} \hat{\chi}_1\, d\zeta = 0, \tag{6.7}$$

since $\hat{\psi}_1 = 0$ at either end of the range of integration. It follows that

$$\int_0^{(\alpha^2-1)^{\frac{1}{2}}} \hat{\chi}_1\, d\zeta \tag{6.8}$$

is a linear function of υ and, in consequence, cannot tend to two different finite values as $\upsilon \to \pm\infty$. The discontinuity in χ cannot therefore be an order of magnitude greater than $BR^{-\frac{1}{12}}$.

Secondly, suppose that χ is continuous but $\partial\chi/\partial\rho$ is not. Then

$$\chi \to B, \quad \partial\chi/\partial\upsilon \sim AR^{-\frac{1}{12}} \quad \text{as} \quad \upsilon \to \infty, \tag{6.9a}$$

$$\chi \to B, \quad \partial\chi/\partial\upsilon \sim BR^{-\frac{1}{21}} \quad \text{as} \quad \upsilon \to -\infty, \tag{6.9b}$$

where we may take A, B to be bounded as $R \to \infty$. In the inner shear layer we write

$$\chi = B + B\hat{\chi}_2 R^{\gamma} + \ldots, \quad \psi = B\hat{\psi}_2 R^{\gamma-\frac{1}{3}} + \ldots, \tag{6.10}$$

where $\gamma = -\frac{1}{21}$, unless $B \ll R^{-\frac{1}{28}}$ and then $BR^{\gamma} \sim R^{-\frac{1}{12}}$. Again, the boundary condition at $\zeta = (\alpha^2-1)^{\frac{1}{2}}$ reduces to

$$\hat{\psi}_2 = 0 \tag{6.11}$$

whatever the value of γ. The boundary condition at $\zeta = 0$, $\rho < 1$ becomes

$$1 + R^{\gamma}\hat{\chi}_2 = 2[-2\upsilon]^{\frac{1}{4}}\hat{\psi}_2 R^{\gamma+\frac{1}{12}} \tag{6.12}$$

and also reduces to (6.11) since $\gamma + \frac{1}{12} > 0$ from (6.9). The argument leading to the exclusion of a discontinuity in χ may now be repeated to show that the leading terms in $\partial\chi/\partial\rho$ must be continuous at the inner edges of the outer shear layers.

For parallel argument to succeed in excluding a discontinuity in $\partial^2\chi/\partial\rho^2$ the necessary discontinuity must have an order of magnitude $\gg R^{\frac{1}{2}}$, which is not satisfied in the present problem.

For an inner layer to be possible therefore we must choose the unknown constants A and B to satisfy

$$1 - A = B \tag{6.13}$$

(continuity of χ at $\rho = 1$),

$$\frac{AR^{\frac{1}{4}}\alpha^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{8}}} = B\mathscr{F}'(0)\left[\frac{R^2}{2(\alpha^2-1)^2}\right]^{\frac{1}{7}} + 2^{\frac{1}{7}}\alpha^{\frac{1}{2}}(\alpha^2-1)^{-\frac{13}{28}}R^{\frac{3}{14}}\mathscr{G}'(0) + \text{higher-order terms} \tag{6.14}$$

(continuity of $\partial\chi/\partial\rho$ at $\rho = 1$), i.e.

$$B = \frac{\alpha^{\frac{1}{4}}2^{\frac{1}{7}}}{(\alpha^2-1)^{\frac{5}{6}}\mathscr{F}'(0)}R^{-\frac{1}{28}} + \ldots, \tag{6.15}$$

from which A follows. The implication of (6.15) is that practically the whole of the change in azimuthal velocity occurs when $\rho > 1$, specifically $\rho - 1 \sim R^{-\frac{1}{4}}$, and very little more is done, when $\rho < 1$, than to flatten out the vertical tangent in the graph of χ outside the shear layer. In order to determine the leading terms in the inner layer write

$$\chi = B + \{(AR^{-\frac{1}{12}}\alpha^{\frac{1}{4}})/(\alpha^2-1)^{\frac{3}{8}}\}\upsilon + R^{\frac{1}{3}}\overline{\chi}, \quad \psi = \overline{\psi}. \tag{6.16}$$

Then $\overline{\psi}$, $\overline{\chi}$ are of the same order of magnitude, which turns out to be $R^{-\frac{19}{42}}$. They satisfy the differential equations

$$\frac{\partial^4\overline{\psi}}{\partial\upsilon^4} = 2\frac{\partial\overline{\chi}}{\partial\zeta}, \quad \frac{\partial^2\overline{\chi}}{\partial\upsilon^2} = -2\frac{\partial\overline{\psi}}{\partial\zeta} \tag{6.17}$$

and the following boundary conditions:

(1) at $\zeta = (\alpha^2-1)^{\frac{1}{2}}$

$$\overline{\psi} = \frac{1}{2R^{\frac{1}{2}}}\left[\frac{\alpha^2}{\alpha^2-1}\right]^{\frac{1}{4}} + O(R^{-\frac{15}{28}}); \tag{6.18a}$$

(2) at $\zeta = 0$ $\quad \overline{\psi} = 0 \quad \text{if} \quad \upsilon > 0,$

$$\overline{\psi} = \{BR^{-\frac{5}{12}}/2(-2\upsilon)^{\frac{1}{4}}\} + O(R^{-\frac{1}{2}}) \quad \text{if} \quad \upsilon < 0; \tag{6.18b}$$

(3) as $\upsilon \to \infty$

$$\frac{\partial^2\overline{\chi}}{\partial\upsilon^2} \to \frac{R^{-\frac{1}{2}}\alpha^{\frac{1}{2}}}{(\alpha^2-1)^{\frac{3}{4}}} + \ldots, \quad \overline{\psi} \to \frac{\alpha^{\frac{1}{2}}\zeta R^{-\frac{1}{2}}}{2(\alpha^2-1)^{\frac{3}{4}}} + \ldots; \tag{6.18c}$$

(4) as $\upsilon \to -\infty$

$$(-2\upsilon)^{\frac{1}{4}}\frac{\partial^2\overline{\chi}}{\partial\upsilon^2} \to \frac{BR^{-\frac{5}{12}}}{(\alpha^2-1)^{\frac{1}{2}}} + \ldots, \quad (-2\upsilon)^{\frac{1}{4}}\overline{\psi} \to \frac{BR^{-\frac{5}{12}}}{(\alpha^2-1)^{\frac{1}{2}}}[(1-\alpha^2)^{\frac{1}{2}} - \zeta] + \ldots. \tag{6.18d}$$

Consequently, on expanding $\overline{\chi}$, $\overline{\psi}$ in descending powers of R the leading terms are

$$\overline{\chi} = R^{-\frac{19}{42}}\overline{\chi}_1 + \ldots, \quad \overline{\psi} = R^{-\frac{19}{42}}\overline{\psi}_1 + \ldots. \tag{6.19}$$

Here $\overline{\chi}_1$, $\overline{\psi}_1$ satisfy (6.17) together with the boundary conditions

$$\left.\begin{array}{l}\overline{\psi}_1 = 0 \quad \text{at} \quad \zeta = (\alpha^2-1)^{\frac{1}{2}},\\ \overline{\psi}_1 = 0 \quad \text{at} \quad \zeta = 0, \quad \upsilon > 0 \quad \text{and} \quad \overline{\psi}_1 = (-2\upsilon)^{-\frac{1}{4}}C \quad \text{at} \quad \zeta = 0, \quad \upsilon < 0,\\ \overline{\psi}_1 \to 0 \quad \text{as} \quad \upsilon \to \infty \quad \text{and} \quad (-2\upsilon)^{\frac{1}{4}}\overline{\psi}_1 \to C\dfrac{(1-\alpha^2)^{\frac{1}{2}} - \zeta}{(1-\alpha^2)^{\frac{1}{2}}} \quad \text{as} \quad \upsilon \to -\infty,\end{array}\right\} \tag{6.20}$$

where

$$C = \tfrac{1}{2}BR^{\frac{1}{28}}, \tag{6.21}$$

and is independent of R. The formal solution for $\overline{\psi}_1$, $\overline{\chi}_1$ is

$$\overline{\psi}_1 = \frac{(-\frac{1}{4})!\,C}{2^{\frac{1}{4}}.2\pi}\int_{-\infty}^{\infty}\frac{\sinh\frac{1}{2}\omega^3(\sqrt{(1-\alpha^2)}-\zeta)}{\sinh\frac{1}{2}\omega^3\sqrt{(1-\alpha^2)}}\,\frac{e^{i\omega\upsilon}\,d\omega}{(0-i\omega)^{\frac{3}{4}}}, \tag{6.22a}$$

$$\bar{\chi}_1 = -\frac{(-\frac{1}{4})!\,C}{2^{\frac{1}{4}}.\,2\pi}\int_{-\infty}^{\infty}\frac{\cosh\frac{1}{2}\omega^3(\sqrt{(1-\alpha^2)}-\zeta)}{\sinh\frac{1}{2}\omega^3\sqrt{(1-\alpha^2)}}\,\frac{\omega\, e^{i\omega v}\,d\omega}{(0-i\omega)^{\frac{3}{4}}}, \tag{6.22b}$$

in which $(0-i\omega)^{\frac{3}{4}}$ is a regular function of ω, except on the negative imaginary axis, and equal to $e^{-\frac{3}{8}\pi i}\omega^{\frac{3}{4}}$ when ω is real and positive. It is not difficult to evaluate these integrals in terms of the residues at the poles of $\sinh[\frac{1}{2}\omega^3\sqrt{(1-\alpha^2)}]$ and, when $v < 0$, an integral along the two sides of the negative imaginary axis. Further terms in the expansion of $\bar{\psi}_1$, $\bar{\chi}_1$ in descending powers of R may now be worked out in principle. One special feature of (6.22), however, calls for careful consideration, namely the behaviour of $\bar{\psi}_1$, $\bar{\chi}_1$ near $\zeta = v = 0$. If for example we set $\zeta = 0$ and suppose v is small and negative, (6.22) reduces to

$$\bar{\psi}_1 \sim (-v)^{-\frac{1}{4}}, \quad \bar{\chi}_1 \sim (-v)^{-\frac{5}{4}}. \tag{6.23}$$

Such behaviour implying that $\bar{\psi}_1 \to \infty$ and $\bar{\chi}_1$ is non-integrable near $v = \zeta = 0$ seems hardly satisfactory at first sight. It will now be shown however that in fact it is in line with the match that the shear layer has to make with the form taken by the Ekman layer in the neighbourhood of $\rho = 1$, $\zeta = 0$. For near $v = \zeta = 0$

$$\bar{\psi}_1 = (C/\zeta^{\frac{1}{12}})F_1(\Phi), \quad \bar{\chi}_1 = (C/\zeta^{\frac{5}{12}})\,F_2(\Phi), \tag{6.24}$$

where $\Phi = v/\zeta^{\frac{1}{3}}$ and

$$F_1(\Phi) = \frac{(-\frac{1}{4})!}{2^{\frac{1}{4}}\pi}\int_0^{\infty}\frac{\cos(\omega\Phi+\frac{3}{8}\pi)}{\omega^{\frac{3}{4}}}\,e^{-\frac{1}{2}\omega^3}\,d\omega, \tag{6.25a}$$

$$F_2(\Phi) = -\frac{(-\frac{1}{4})!}{2^{\frac{1}{4}}\pi}\int_0^{\infty}\cos(\omega\Phi+\tfrac{3}{8}\pi)\,\omega^{\frac{1}{4}}\,e^{-\frac{1}{2}\omega^3}\,d\omega. \tag{6.25b}$$

It may be shown that, as $\Phi \to -\infty$, $F_1(\Phi) \sim (-\Phi)^{-\frac{1}{4}}$ and $F_2(\Phi) \sim (-\Phi)^{-\frac{5}{4}}$ in agreement with (6.23). Now the assumed expansion of χ, ψ in descending powers of R ((6.16) and (6.19)) is no longer justified when the later terms of the series are the same order of magnitude as the earlier ones. In the case of χ this occurs when

$$B \sim R^{\frac{1}{3}}\bar{\chi},$$

i.e.

$$R^{-\frac{1}{28}} \sim R^{\frac{1}{3}-\frac{19}{42}}\zeta^{-\frac{5}{12}}F_2(\Phi). \tag{6.26}$$

Taking $\Phi = O(1)$ this implies that

$$\zeta \sim R^{-\frac{1}{5}}, \quad \rho - 1 \sim R^{-\frac{2}{5}}, \tag{6.27}$$

exactly the order of magnitude of ζ, ρ when the Ekman-layer equations change character. The orders of magnitude are unaffected if we suppose $|\Phi| \gg 1$ for then, when (6.27) holds

$$\psi \sim R^{-\frac{1}{28}-\frac{2}{5}}, \quad \chi \sim B \sim R^{-\frac{1}{28}} \tag{6.28}$$

so that the ratio of the orders of magnitude of ψ, χ is as required in (3.15). We conclude that in principle a match may be effected between the shear layer and the modified Ekman layer and that, using the notation of (3.16),

$$\left.\begin{aligned} \chi^* &\approx 2CR^{-\frac{1}{28}}\left[1+\frac{1}{\zeta^{*\frac{5}{12}}}F_2\left(\frac{y^*}{\zeta^{*\frac{1}{3}}}\right)+\ldots,\right. \\ \psi^* &\approx 2CR^{-\frac{1}{28}}\left[\frac{1}{\zeta^{*\frac{1}{12}}}F_1\left(\frac{y^*}{\zeta^{*\frac{1}{3}}}\right)+\ldots,\right. \end{aligned}\right\} \tag{6.29}$$

when y^*, ζ^* are large. Since the governing equations in the limiting Ekman layer and the shear layer are the same it is reasonable to expect [although it has not been comprehensively checked] that further terms in the expansion of $\bar{\chi}$, $\bar{\psi}$ in descending powers of R will, when (6.27) is satisfied, lead to further terms in (6.29) of order $R^{-\frac{1}{28}}$ at most. The behaviour of the shear layer near $\rho = 1$, $\zeta = 0$ is accordingly acceptable.

7. The terminal form of the Ekman layer on the inner sphere

The differential equations satisfied by χ^*, ψ^* are set out in (3.18) and one set of boundary conditions are given in (6.29). Since the boundary of the sphere is $\lambda = 0$ (3.15), in terms of y^*, ζ^* it is given by

$$y^* = -\tfrac{1}{2}\zeta^{*2} \tag{7.1}$$

and here

$$\psi^* = \partial\psi^*/\partial\lambda = \chi^* = 0. \tag{7.2}$$

Further on the line of symmetry $\zeta^* = 0, y^* > 0$

$$\psi^* = \partial\chi^*/\partial\Theta = \partial\chi^*/\partial\zeta^* = 0. \tag{7.3}$$

The boundary conditions (6.29) only hold when we are well away from the sphere, strictly when ζ^* is large and $\zeta^* \sim y^{*3}$. Near the sphere χ^*, ψ^* must match with the Ekman solution as $\Theta \to \infty$ so that

$$\chi^* \to 2CR^{-\frac{1}{28}}[1 - e^{-\eta^*}\cos\eta^*], \tag{7.4a}$$

$$2\Theta^{\frac{1}{2}}\psi^* \to 2CR^{-\frac{1}{28}}[1 - e^{-\eta^*}(\cos\eta^* + \sin\eta^*)] \tag{7.4b}$$

as $\Theta \to \infty$ for fixed $\lambda\Theta^{\frac{1}{2}} = \eta^*$. These two sets of conditions (6.29) and (7.4) match when η^* is large and $y^*/\zeta^{*\frac{1}{3}}$ is large and negative.

It appears that the boundary conditions set here [(6.29), (7.2), (7.3), (7.4)] are consistent and sufficient to determine χ^*, ψ^* in principle although no proof is available. It may be, however, that not all are necessary for it seems likely that (6.29) is implied by (7.4).

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